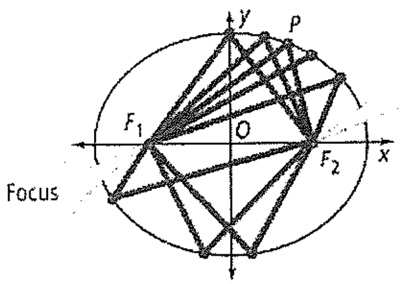
Ellipses

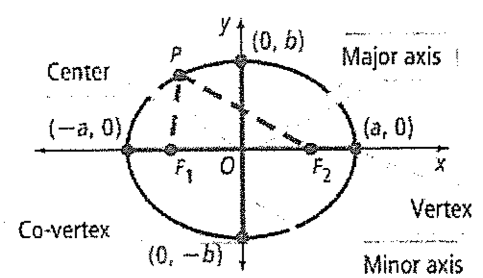
Graph



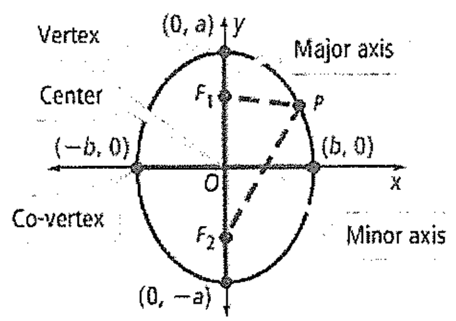
$PF_1 + PF_2 = k$, where $k > F_1F_2$.

Ellipse: a set of all points P in a plane such that the sum of the distances from P to two fixed points F_1 and F_2 is a constant.

Foci of an Ellipse: One of the two fixed points.



Horizontal Ellipse



Vertical Ellipse

Major Axis: The segment that contains the foci and has end points on the ellipse.

Center of the ellipse: The midpoint of the major axis.

Minor Axis: Perpendicular to the major axis the center.

Vertices: The endpoints of the major axis

Co-vertices: The endpoints of the minor axis.

The standard form of the equation of an ellipse with center (h,k) is:

$$\boxed{\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1}$$

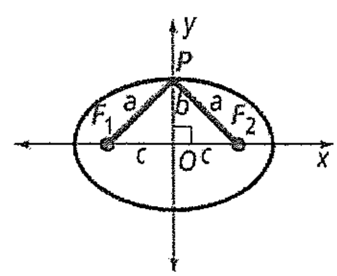
Key Concept **Properties of Ellipses with Center (0, 0)**

	Horizontal Ellipses	Vertical Ellipses
Standard Equation	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, a > b > 0$	$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1, a > b > 0$
Major Axis	horizontal	vertical
Vertices	$(\pm a, 0)$	$(0, \pm a)$
Co-vertices	$(0, \pm b)$	$(\pm b, 0)$
Foci	$(\pm c, 0)$ on x-axis	$(0, \pm c)$ on y-axis

The length of the major axis is $2a$ and the length of the minor axis is $2b$.
For any point P on an ellipse, $PF_1 + PF_2 = 2a$.

Key Concept	***Equations of Ellipses with Centers at (h, k)***	
Standard Form of Equation	$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$	$\frac{(y-k)^2}{a^2} + \frac{(x-h)^2}{b^2} = 1$
Direction of Major Axis	horizontal	vertical
Foci	$(h \pm c, k)$	$(h, k \pm c)$

Since the co-vertex $P(0, b)$ is on the ellipse, $PF_1 + PF_2 = 2a$. If you denote the distance from each focus to the center of the ellipse by c, then a, b, and c are the lengths of the sides of a right triangle, as shown in the ellipse at the right. Thus, the distances from the center to each vertex, to each co-vertex, and to each focus are related by the Pythagorean Theorem: $a^2 = b^2 + c^2$.



If $(\pm a, 0)$, $(0, \pm b)$, and $(\pm c, 0)$ are the vertices, the co-vertices, and the foci of an ellipse, respectively,

$$c^2 = a^2 - b^2$$

Example 11: Write the equation for an ellipse in standard form centered at the origin with vertex (-6,0) and co-vertex (0,3).

a

b

$$\frac{x^2}{36} + \frac{y^2}{9} = 1$$

Example 12: What are the foci of the ellipse with the equation $25x^2+9y^2=225$? Graph the foci and the ellipse.

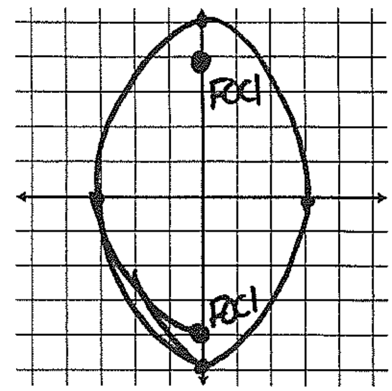
$$\frac{25x^2}{225}+\frac{9y^2}{225}=\frac{225}{225}$$

$$\frac{x^2}{9}+\frac{y^2}{25}=1$$

$25>9$ so the major axis is vertical

$c^2=a^2-b^2=25-9=16$ $c=\pm 4$

foci are $(0,-4)$ and $(0,4)$



Example 13:

Whispering Gallery A room with an elliptical ceiling (called an *ellipsoid,* since it is 3-dimensional) forms a "whispering gallery." Thanks to the reflective property of the ellipse, a whispered message at one focus can be heard clearly by someone standing across the room at the other focus. If the elliptical ceiling has a major axis of 120 feet and a minor axis of 72 feet, how far apart are the foci?

length of major axis = $2a$

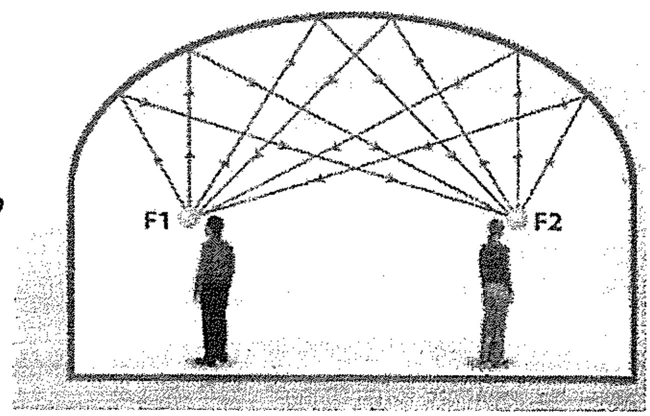
so $2a=120$ $a=60$

minor axis = $2b$ so $2b=72$ $b=36$

$c^2=a^2-b^2=60^2-36^2$

$c=\sqrt{60^2-36^2}=48$

so foci are $2c=96$ft



Example 14:

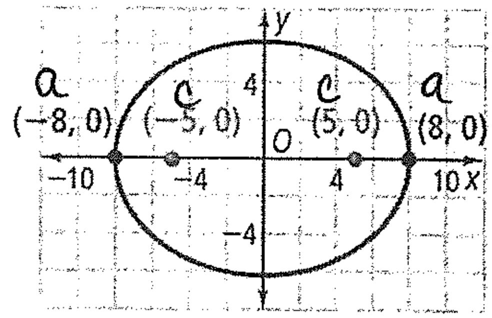
What is the standard form equation of the ellipse shown?

$c^2 = a^2 - b^2$

$25 = 64 - b^2$

$b^2 = 64 - 25 = 39$

$\frac{x^2}{64} + \frac{y^2}{39} = 1$



Identify the center, vertices, co-vertices, foci, length of the major and minor axes.

1. $\frac{x^2}{64} + \frac{(y-6)^2}{121} = 1$

Center $(0, 6)$
Vertices $(0, 17), (0, 5)$
Co-vertices $(8, 6)$ $(-8, 6)$
Foci $(0, 6+\sqrt{57}), (0, 6-\sqrt{57})$
major axis = 22
minor axis = 16

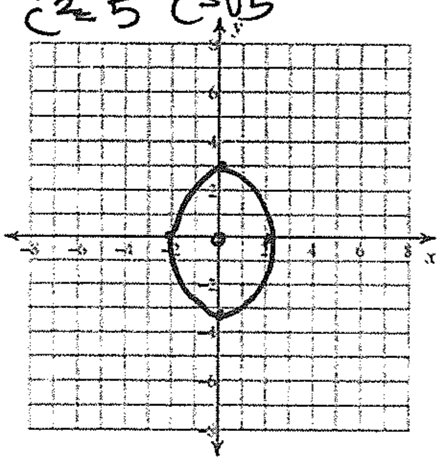
2. $\frac{(x+5)^2}{81} + \frac{(y-1)^2}{144} = 1$

Center $(-5, 1)$
Vertices $(-5, 13), (-5, -11)$
Co-Vertices $(4, 1), (-14, 1)$
Foci $(-5, 1+3\sqrt{7}), (-5, 1-3\sqrt{7})$
maj. axis = 24
minor axis = 18

Identify the center, vertices, co-vertices, foci, length of the major and minor axes and graph the ellipse.

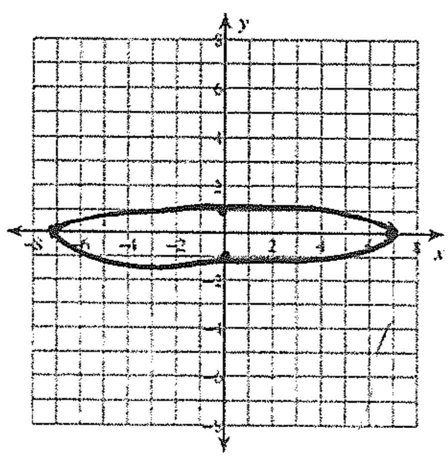
3. $\frac{x^2}{4} + \frac{y^2}{9} = 1$

center $(0, 0)$
major axis = 6
minor axis = 4
vertices ~~$(\pm 3, 0)$~~ $(0, \pm 3)$
Co-vertices $(\pm 2, 0)$
$a^2 = b^2 + c^2$
$9 = 4 + c^2$
$c^2 = 5$ $c = \sqrt{5}$
Foci $(0, \pm\sqrt{5})$



4. $\frac{x^2}{49} + y^2 = 1$

Center $(0, 0)$
major axis = 14
minor axis = 2

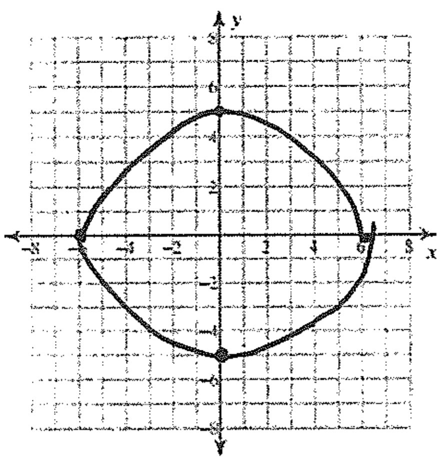


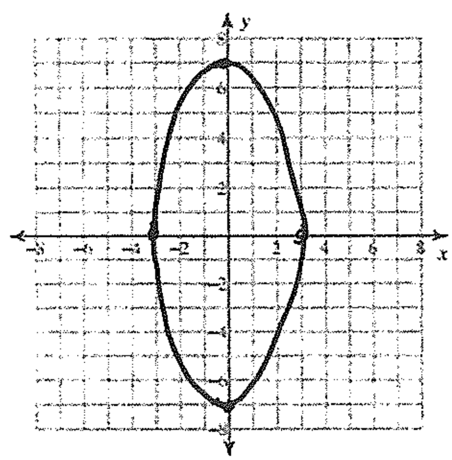
5. $\frac{x^2}{36} + \frac{y^2}{25} = 1$

Center (0,0)
maj axis = 12
minor axis = 10

6. $\frac{x^2}{9} + \frac{y^2}{49} = 1$

Center (0,0)
major axis = 14
minor axis = 6





Use the information provided to write the standard form equation of each ellipse.

7. Foci: $(\sqrt{17}, 0)$, $(-\sqrt{17}, 0)$
   +c -c
   Endpoints of major axis: $(9, 0)$, $(-9, 0)$
   a a

$\frac{x^2}{81} + \frac{y^2}{64} = 1$

$a^2 = b^2 + c^2$

$81 = b^2 + 17$

$b^2 = 64 \quad b = 8$

8. Foci: $(\sqrt{115}, 0)$, $(-\sqrt{115}, 0)$
   Endpoints of major axis: $(\sqrt{195}, 0)$, $(-\sqrt{195}, 0)$

$\frac{x^2}{195} + \frac{y^2}{80} = 1$

9. vertical
   Center: $(6, -5)$ } 12 a
   Vertex: $(6, 7)$
   Focus: $(6, -5 - 6\sqrt{3})$ $(h, k - c)$

$\frac{(x-6)^2}{36} + \frac{(y+5)^2}{144} = 1$

$108 = 144 - b^2$

$b^2 = 144 - 108 = 36$

h k
10. Center: $(7, -10)$
    Vertex: $(-6, -10)$ a k
    Co-vertex: $(7, -17)$ h b

$\frac{(x-7)^2}{169} + \frac{(y+10)^2}{49}$

11. Center: $(1, -7)$
    Vertex: $(1, 1)$
    $c^2 = 55$

$\frac{(x-1)^2}{9} + \frac{(y+7)^2}{64} = 1$

12. Center: $(4, 0)$
    Focus: $(4, 3\sqrt{7})$
    Width: 18

$\frac{(x-4)^2}{81} + \frac{y^2}{144} = 1$

Hyperbolas

Hyperbola: The set of all points P in a plane such that the absolute value of the difference between the distances from P to two fixed points F_1 & F_2 is constant

The standard form of the equation of a parabola with vertex (h,k)

$$\boxed{\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1}$$

Focus of the hyperbola: The two fixed points on traverse axis

Vertex: the turning point of each branch of the hyperbola

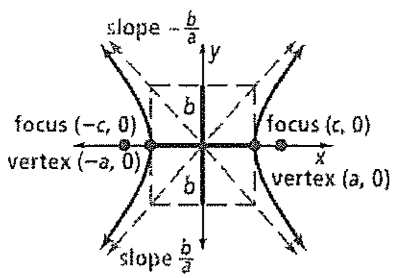
Transverse axis: the segment connecting the two vertices

Axis of symmetry: the axis on which the traverse axis lies.

Center of the hyperbola: midpoint between the two vertices.

Key Concept Properties of Hyperbolas with Center (0, 0)

Horizontal Hyperbola



Equation: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

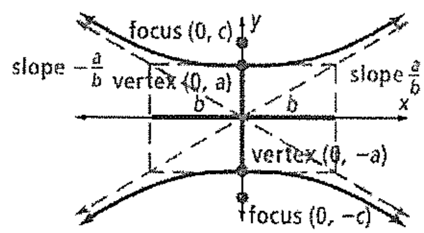
Transverse axis: Horizontal

Vertices: $(\pm a, 0)$

Foci: $(\pm c, 0)$, where $c^2 = a^2 + b^2$

Asymptotes: $y = \pm\frac{b}{a}x$

Vertical Hyperbola



Equation: $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$

Transverse axis: Vertical

Vertices: $(0, \pm a)$

Foci: $(0, \pm c)$, where $c^2 = a^2 + b^2$

Asymptotes: $y = \pm\frac{a}{b}x$

Key Concept Equations of Hyperbolas with Centers at (h, k)

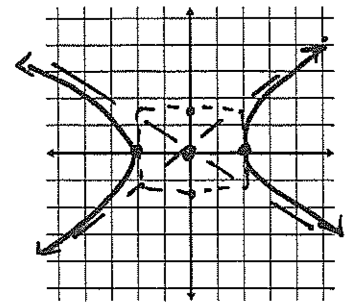
Standard Form of Equation	$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$	$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$
Direction of Transverse Axis	horizontal	vertical
Equations of Asymptotes	$y - k = \pm\frac{b}{a}(x-h)$	$y - k = \pm\frac{a}{b}(x-h)$

Example 15: A hyperbola centered at (0,0) has vertices (±4,0) and one focus (5,0) write the standard form equation of the hyperbola and graph it.

$c^2 = a^2 + b^2$
$b^2 = c^2 - a^2$
$b^2 = 25 - 16$
$b^2 = 9$
$b = 3$

$$\frac{x^2}{4^2} - \frac{y^2}{3^2} = 1$$

$$\frac{x^2}{16} - \frac{y^2}{9} = 1$$

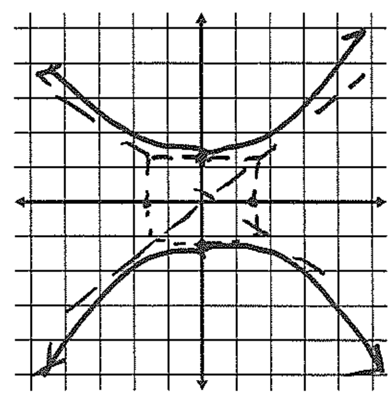


Example 16: What are the vertices, foci and asymptotes of the hyperbola with equation $9y^2 - 7x^2 = 63$

Graph the hyperbola.

$$\frac{9y^2}{63} - \frac{7x^2}{63} = 1$$

$$\frac{y^2}{7} - \frac{x^2}{9} = 1$$



Identify the vertices, foci, and direction of opening of each.

1. $\frac{y^2}{25} - \frac{x^2}{16} = 1$

vertices $(0, \pm 5)$
foci $(0, \pm\sqrt{41})$
opens up/ down

2. $\frac{x^2}{121} - \frac{y^2}{36} = 1$

Vertices $(\pm 11, 0)$
foci $(\pm\sqrt{157}, 0)$
opens L-R

3. $\frac{(x+2)^2}{169} - \frac{(y+8)^2}{4} = 1$

Vertices: $(11, -8)$
$(-15, -8)$
foci: $(-2+\sqrt{173}, -8)$
$(-2-\sqrt{173}, -8)$

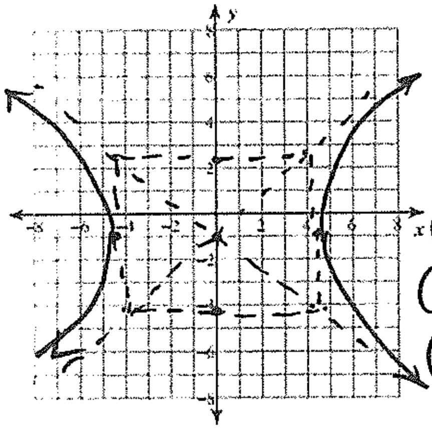
opens L-R

4. $\frac{(y+8)^2}{36} - \frac{(x+2)^2}{25} = 1$

Vertices: $(-2,-2)$ $(-2,-14)$
Foci: $(-2, -8+\sqrt{61}$, $(-2, -8-\sqrt{61})$
Opens up/down

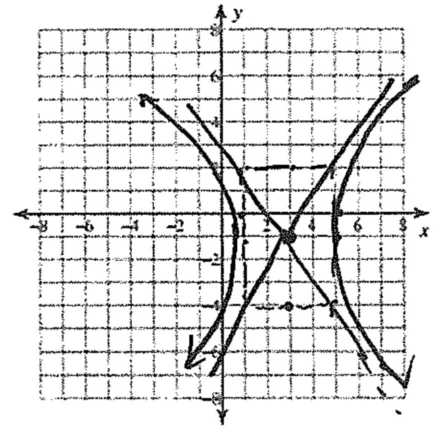
Identify the vertices and foci of each. Then sketch the graph.

5. $\dfrac{x^2}{20} - \dfrac{(y+1)^2}{10} = 1$



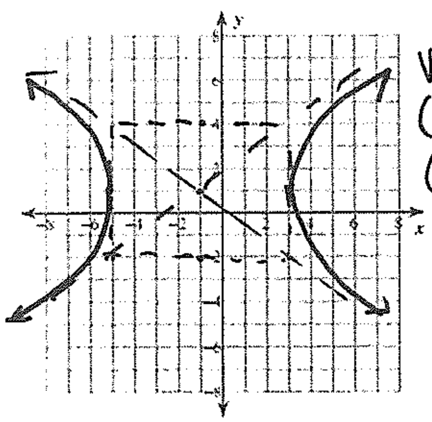
Vertices
$(2\sqrt{5}, -1)$
$(-2\sqrt{5}, -1)$
Foci
$(\sqrt{30}, -1)$
$(-\sqrt{30}, -1)$

6. $\dfrac{(x-3)^2}{4} - \dfrac{(y+1)^2}{9} = 1$



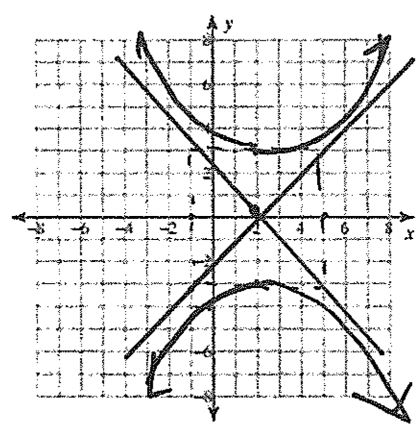
Vertices: $(5, -1)$
$(1, -1)$
Foci $(3+\sqrt{13}, -1)$
$(3-\sqrt{13}, -1)$

7. $\dfrac{(y-1)^2}{9} - \dfrac{(x+1)^2}{16} = 1$



Vertices:
$(-1, 4)$
$(-1, -2)$
foci $(-1, 6)$
$(-1, 4)$

8. $\dfrac{y^2}{9} - \dfrac{(x-2)^2}{9} = 1$



Vertices: $(2, 3)$
$(2, -3)$
foci $(2, 3\sqrt{2})$
$(2, -3\sqrt{2})$

Use the information provided to write the standard form equation of each hyperbola.

9. $-x^2 + y^2 - 18x - 14y - 132 = 0$

$$\frac{(y-7)^2}{100} - \frac{(x+9)^2}{100} = 1$$

10. $9x^2 - 4y^2 - 90x + 32y - 163 = 0$

$$\frac{(x-5)^2}{36} - \frac{(y-4)^2}{81} = 1$$

11. Vertices: $(8, 14)$, $(8, -10)$
Conjugate Axis is 6 units long

$$\frac{(y-2)^2}{144} - \frac{(x-8)^2}{9} = 1$$

12. Vertices: $(4, 9+\sqrt{30})$, $(4, 9-\sqrt{30})$
Conjugate Axis is $2\sqrt{195}$ units long

$$\frac{(y-9)^2}{30} - \frac{(x-4)^2}{195} = 1$$